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# REMARKS ON ISOMORPHISMS OF REGRESSIVE TRANSFORMATION SEMIGROUPS

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For a (finite or infinite) set $X$, let $T(X)$ be *the full transformation semigroup on* $X$, i.e. the set of all maps from $X$ to $X$, the semigroup operation being composition of maps.

When $X$ is a partially ordered set, we let

$$T_{RE}(X) = \{ f \in T(X) \mid f(x) \leq x \text{ for all } x \in X \},$$

$$T_{OP}(X) = \{ f \in T(X) \mid f(x) \leq f(y) \text{ if } x \leq y \text{ for } x, y \in X \}.$$

Then, both of them are subsemigroups of $T(X)$ with the identity $id_{T(X)}$. We call $T_{RE}(X)$ *the full regressive transformation semigroup on* $X$, and $T_{OP}(X)$ *the full order-preserving transformation semigroup on* $X$.

Recently, some interesting results on $T_{RE}(X)$ have been obtained (cf. [1], [4], [5]).

It is known that, for partially ordered sets $X$, $Y$, if $T_{OP}(X)$ and $T_{OP}(Y)$ are isomorphic as semigroups, then $X$ and $Y$ are isomorphic or anti-isomorphic as ordered sets (see [3], Theorem V.8.9).

It is natural to ask whether the above result holds or not for regressive transformation semigroups. In general, it does not hold. However, we obtain a necessary and sufficient condition on partially ordered sets $X$ and $Y$ for $T_{RE}(X)$ and $T_{RE}(Y)$ to be isomorphic.

Umar showed in [6] that, when $X$ and $Y$ are totally ordered sets, any idempotent in $T_{RE}(X)$ whose image is an order-ideal is mapped to an idempotent in $T_{RE}(Y)$ with the same property by isomorphisms from $T_{RE}(X)$ to $T_{RE}(Y)$, and he considered the above problem through this result. If the result holds even if "an order-ideal" in it is replaced by "a principal order-ideal", then it can be shown that if $T_{RE}(X) \cong T_{RE}(Y)$ as semigroups then $X \cong Y$ as ordered sets. At the present time, this is unsolved.

In here, we achieve our purpose by showing that any idempotent of defect 1 in $T_{RE}(X)$ is mapped to an idempotent of defect 1 in $T_{RE}(Y)$ by isomorphisms from $T_{RE}(X)$ to $T_{RE}(Y)$, where the defect of $\alpha$ in $T_{RE}(X)$ means the cardinality of the set of idempotents in $X$ which do not belong to the image of $\alpha$.

For partially ordered set $X$, an element in $X$ is said to be *isolated* if it is incomparable with every element in $X$ except itself. Let $Is(X)$ be the set of all isolated elements in $X$. Then, it is easy to see that $T_{RE}(X)$ and $T_{RE}(X \setminus Is(X))$ are isomorphic. Therefore, we may assume that every partially ordered set, treated in this paper, does not contain any isolated elements.

Let $X$ be a partially ordered set under the order relation $\leq$.

For $a \in X$, the set of (resp. *strict*) *upper bounds* of $a$ is denoted by $U(a)$ (resp. $SU(a)$), i. e.

$$U(a) = \{x \in X \mid x \geq a\} \quad \text{and} \quad SU(a) = \{x \in X \mid x > a\},$$

and the set of all minimal elements in $X$ is denoted by $Min(X)$,

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This is an abstract and the details will be published in Semigroup Forum.

$b$, we have that $j'(j(a, b), k(a, b)) = a$ and $k'(j(a, b), k(a, b)) = b$.

**Lemma 2.** (1) *$k(a, b) = k(c, d)$ if and only if $b = d$.*
(2) *If $a < c < b$, then $k(a, c) = j(c, b)$ and $j(a, b) = j(a, c)$.*
(3) *$j(a, b) = j(c, d)$ if and only if $a = c$.*

***Proof.*** (1) It is easy to see that

$$b = d \Leftrightarrow \lambda^d{}_c \circ \lambda^b{}_a = \lambda^b{}_a \Leftrightarrow \lambda^{k(c,d)}{}_{j(c,d)} \circ \lambda^{k(a,b)}{}_{j(a,b)} = \lambda^{k(a,b)}{}_{j(a,b)} \Leftrightarrow k(a, b) = k(c, d).$$

This assertion means that $k(a, b)$ depends only on $b$.

(2) The proof is omitted.

(3) To show the assertion, we need that $X$ and $Y$ are adjusted. Let $a = c$. If $a$ is not minimal in $X$, then $e < a$ for some $e \in X$. From (2), we have that $j(a, b) = k(e, a) = j(a, d) = j(c, d)$.

If $a$ is minimal in $X$, then $b$ and $d$ are connected in $SU(a)$, since $X$ is adjusted, so that there exist $e_1, e_2, \dots, e_n \in SU(a)$ such that $b = e_1 \leq^s e_2 \leq^s \dots \leq^s e_n = d$. Since $e_i$ and $e_{i+1}$ are comparable, by (2) we have that $j(a, e_i) = j(a, e_{i+1})$ $(i = 1, 2, \dots, e_{n-1})$. Thus, we have that $j(a, b) = j(a, d) = j(c, d)$.

Let $j(a, b) = j(c, d)$. If we apply the above fact to $j'$, then we have that $a = j'(j(a, b), k(a, b)) = j'(j(c, d), k(c, d)) = c$.

This assertion means that $j(a, b)$ depends only on $a$.

We write $j(a, b) = j(a)$ and $k(a, b) = k(b)$ for $a, b \in X$ with $a < b$. In this case, if $a$ is maximal in $X$, then $j(a)$ is undefined, and if $b$ is minimal in $X$, then $k(b)$ is undefined. Since $j(a) < k(b)$ if $a < b$, we have that if $a$ is not maximal in $X$, then neither is $j(a)$ in $Y$. By (2) of Lemma 2, if $c$ is neither maximal nor minimal in $X$, then $j(c) = k(c)$.

Similarly, we write $j'(a', b') = j'(a')$ and $k'(a', b') = k'(b')$ for $a', b' \in Y$ with $a' < b'$. Then, we have that $j'(j(a)) = a$, $k'(k(b)) = b$, $j(j'(a')) = a'$ and $k(k'(b')) = b'$.

Let $a$ be maximal in $X$. Then, we can show that $k(a)$ is maximal in $Y$.

Define a map $h : X \to Y$ by $h(a) = j(a)$ if $a$ is not maximal in $X$, and $h(a) = k(a)$ if $a$ is maximal in $X$. Then, we can show that the $h$ is an order-isomorphism of $X$ onto $Y$.

Since any totally ordered set is clearly adjusted, we obtain :

**Corollary 3.**
*Let $X$ and $Y$ be totally ordered sets. Then, $T_{RE}(X)$ and $T_{RE}(Y)$ are isomorphic as semigroups if and only if $X$ and $Y$ are isomorphic as ordered sets.*

Let $X$, $Y$ be partially ordered sets. From Theorem 1 and Theorem 2, we have that

$$T_{RE}(X) \cong T_{RE}(Y) \Leftrightarrow T_{RE}(A(X)) \cong T_{RE}((A(Y)) \Leftrightarrow A(X) \cong A(Y).$$

Thus, we obtain the following main theorem :

**Theorem 4.**
*Let $X$ and $Y$ be partially ordered sets. Then, $T_{RE}(X)$ and $T_{RE}(Y)$ are isomorphic as semigroups if and only if their adjusted sets $A(X)$ and $A(Y)$ are isomorphic as ordered sets.*

and the set of all minimal elements in $X$ is denoted by $Min(X)$.

Let $\leq^s$ be *the symmetric relation generated by* $\leq$, i.e. $a \leq^s b$ if and only if $a \leq b$ or $b \leq a$, and let $\leq^e$ be *the equivalent relation generated by* $\leq$, i.e. $a \leq^e b$ if and only if there exist $c_1, c_2, \ldots, c_n \in X$ such that $a = c_1 \leq^s c_2 \leq^s \ldots \leq^s c_n = b$ (see [2], I). In this case, we say that $a$ and $b$ are *connected* in $X$. A subset $Y$ of $X$ is *connected* if every $a, b \in Y$ are connected in $Y$, i. e. there exist $c_1, c_2, \ldots, c_n \in Y$ such that $a = c_1 \leq^s c_2 \leq^s \ldots \leq^s c_n = b$

A partially ordered set $X$ is said to be *adjusted* if it does not contain any minimal elements, or for every $m \in Min(X)$, $SU(m)$ is connected.

**Theorem 1.**
*Let $X$ be a partially ordered set. Then, there exists an adjusted partially ordered set $A$ such that $T_{RE}(A)$ is isomorphic to $T_{RE}(X)$ as semigroups.*

We can construct an adjusted partially ordered set $A(X)$ from $X$ such that $T_{RE}(A(X))$ is isomorphic to $T_{RE}(X)$. In this case, the $A(X)$ is called *the adjusted partially ordered set of* $X$.

**Theorem 2.**
*Let $X$, $Y$ be adjusted partially ordered sets. Then, $T_{RE}(X)$ and $T_{RE}(Y)$ are isomorphic as semigroups if and only if $X$ and $Y$ are isomorphic as ordered sets.*

Suppose that $X$ and $Y$ are isomorphic. Let $h$ be an isomorphism from $X$ onto $Y$. Then, it is easy to show that the map $i : T_{RE}(X) \to T_{RE}(Y), f \to i(f)$ defined by $i(f)(h(x)) = h(y)$ if $f(x) = y$, is an isomorphism.

To show the only if–part, we need two lemmas (Lemmas 1 and 2).

For each pair $a, b \in Z$ with $a < b$, where $Z$ is a partially ordered set, we define $\lambda^b{}_a$ in $T_{RE}(Z)$ by

$$\lambda^b{}_a(b) = a, \quad \lambda^b{}_a(x) = x \text{ if } x \neq b.$$

From now until the end of the proof of Theorem 2, $X$ *and* $Y$ will denote adjusted partially ordered sets, and $i$ will denote an isomorphism from $T_{RE}(X)$ onto $T_{RE}(Y)$.

**Lemma 1.** *For each pair $a, b \in X$ with $a < b$, there exist $a', b' \in Y$ such that $i(\lambda^b{}_a) = \lambda^{b'}{}_{a'}$.*

The assertion can be easily shown by using the following facts :

For $g \in T_{RE}(X)$,

$$\lambda^b{}_a \circ g = \lambda^b{}_a \text{ if and only if } g = id_{T(X)} \text{ or } g = \lambda^b{}_a,$$
$$g \circ \lambda^b{}_a = g \text{ if and only if } g(a) = g(b) \text{ and } a < b.$$

For each pair $a, b \in X$ with $a < b$, the pair $a', b'$ in Lemma 1 is clearly unique. So we write

$$a' = j(a, b) \text{ and } b' = k(a, b), \text{ namely } i(\lambda^b{}_a) = \lambda^{k(a, b)}{}_{j(a, b)}.$$

We similarly have that for each pair $a', b'$ in $Y$ with $a' < b'$, there exist unique elements $j'(a', b')$, $k'(a', b')$ in $X$ such that $i^{-1}(\lambda^{b'}{}_{a'}) = \lambda^{k'(a', b')}{}_{j'(a', b')}$. Then, for each $a$, $b$ in $X$ with $a <$

**Corollary 5.**
*Let X and A be as in Theorem 1. Then A is uniquely determined by X up to isomorphisms.*

We next aim to refine Theorem 2 to the following :

**Theorem 6.**
*Let $X$ and $Y$ be as in Theorem 2, and let $i$ be a semigroup isomorphism from $T_{RE}(X)$ onto $T_{RE}(Y)$. Then, there exists an order isomorphism $h$ from $X$ onto $Y$ such that $h(f(a)) = i(f)(h(a))$ for all $f \in T_{RE}(X)$ and all $a \in X$.*

Let $h$ be an isomorphism from $X$ onto $Y$ determined by $i$ in Theorem 6 as in the proof of Theorem 2. Thus, $i(\lambda^b{}_a) = \lambda^{h(b)}{}_{h(a)}$ for each $a, b \in X$ with $a < b$. We show that this $h$ serves as a desired $h$ in Theorem 6. To show the theorem, again we need two lemmas (Lemmas 3 and 4).

For each $f \in T_{RE}(X)$ and each $a \in X$, we define $f^a$ and $f_a$, as follows :

$f^a(x) = x$ if $x \geq a$, $f^a(x) = f(x)$ otherwise, and $f_a(x) = f(x)$ if $x > a$, $f_a(x) = x$ otherwise.

Then, it is easy to check that $f = f_a \circ \lambda^a{}_{f(a)} \circ f^a$ for all $a \in X$, where $\lambda^a{}_{f(a)} = id_{T(X)}$ if $a = f(a)$.

**Lemma 3.** *For any $f \in T_{RE}(X)$ and any $b, c \in X$ with $c \leq b$,*
(1) *$f(b) = f(c)$ if and only if $i(f)(h(b)) = i(f)(h(c))$. In particular,*
(2) *if $a \leq b$, then $i(f^a)(h(b)) = i(f^a)(h(c))$ implies that $h(b) = h(c)$, and*
(3) *if $b \not> a$, then $i(f_a)(h(b)) = i(f_a)(h(c))$ implies that $h(b) = h(c)$, where $b \not> a$ means that $b \leq a$ or $a$ and $b$ are incomparable.*

From Lemma 3, we have :

**Lemma 4.** *For every $a, b \in X$,*
(1) *if $h(b) \geq h(a)$, then $i(f^a)(h(b)) = h(b)$,*
(2) *if $h(b) \not> h(a)$, then $i(f_a)(h(b)) = h(b)$.*

Since $f = f_a \circ \lambda^a{}_{f(a)} \circ f^a$ for all $a \in X$, and since $h(f(a)) \leq h(a)$ , we have that

$$
\begin{aligned}
i(f)(h(a)) &= i(f_a) \circ i(\lambda^a{}_{f(a)}) \circ i(f^a)(h(a)) = i(f_a) \circ \lambda^{h(a)}{}_{h(f(a))}(h(a)) \\
&= i(f_a) \circ \lambda^{h(a)}{}_{h(f(a))}(h(a)) = i(f_a)(h(f(a))) = h(f(a)) .
\end{aligned}
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